Homework 3: Solutions

2.3.2:

(a) $z = \sqrt{a^2 - x^2 - y^2}$, so the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are:

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}.$$
$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}.$$

At the points $(0,0)$ and $(a/2, a/2)$ we get:

$$\frac{\partial z}{\partial x}(0,0) = 0.$$
$$\frac{\partial z}{\partial y}(0,0) = 0.$$
$$\frac{\partial z}{\partial x}(a/2,a/2) = \frac{-a/2}{\sqrt{a^2 - a^2/4 - a^2/4}} = \frac{-a/2}{\sqrt{a^2/2}} = -\frac{1}{\sqrt{2}}\frac{a}{|a|}.$$
$$\frac{\partial z}{\partial y}(a/2,a/2) = \frac{-a/2}{\sqrt{a^2 - a^2/4 - a^2/4}} = \frac{-a/2}{\sqrt{a^2/2}} = -\frac{1}{\sqrt{2}}\frac{a}{|a|}.$$

(b) $z = \log\sqrt{1+xy} = \frac{1}{2}\log(1+xy)$

$$\frac{\partial z}{\partial x} = \frac{y}{2(1+xy)}.$$
$$\frac{\partial z}{\partial y} = \frac{x}{2(1+xy)}.$$

At the points $(1,2)$ and $(0,0)$ we get:

$$\frac{\partial z}{\partial x}(1,2) = \frac{2}{2\cdot 3} = 1/3.$$
$$\frac{\partial z}{\partial y}(1,2) = \frac{1}{2\cdot 3} = 1/6.$$
$$\frac{\partial z}{\partial x}(0,0) = 0.$$
$$\frac{\partial z}{\partial y}(0,0) = 0.$$

(c) $z = e^{ax}\cos(bx+y)$:

$$\frac{\partial z}{\partial x} = ae^{ax}\cos(bx+y) - be^{ax}\sin(bx+y).$$

$$\frac{\partial z}{\partial y} = -e^{ax}\sin(bx+y).$$

So at the point $(2\pi/b, 0)$ we get:

$$\frac{\partial z}{\partial x}(2\pi/b,0) = ae^{a2\pi/b}\cos(2\pi) - b\sin(2\pi) = ae^{2\pi\frac{a}{b}}.$$

$$\frac{\partial z}{\partial y}(2\pi/b,0) = -ae^{a2\pi/b}\sin(2\pi) = 0.$$

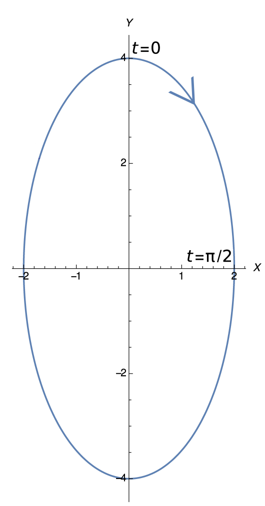
2.3.25:

$f(x,y,z) = x^2 + y^2 - z^2$, so the gradient at (x,y,z) and in particular at $(0,0,1)$ is:

$\nabla f(x,y,z) = (2x, 2y, -2z)$

$\nabla f(0,0,1) = (0,0,-2) = -2\mathbf{k}$

2.4.2: The curve $\mathbf{c}(t) = (2\sin t, 4\cos t), 0 \le t \le 2\pi$ looks like:



2.4.8:

$\mathbf{c}(t) = (\sin 3t)\mathbf{i} + (\cos 3t)\mathbf{j} + 2t^{3/2}\mathbf{k}$, so the velocity vector is:

$\mathbf{c}'(t) = (3\cos 3t)\mathbf{i} + (-3\sin 3t)\mathbf{j} + 3t^{1/2}\mathbf{k}$

2.4.16:

The position of the particle in space is $\mathbf{c}(t) = (6t, 3t^2, t^3)$, so its velocity vector at time t = 0 is:

$\mathbf{c}'(0) = (6, 6t, 3t^2)|_{t=0} = (6, 0, 0)$.

2.4.22:

The particle moving along $\mathbf{c}(t) = (\sin e^t, t, 4 - t^3)$, flies off on tangent at $t = t_0 = 1$. Want its position at time $t = t_1 = 2$, (i.e. 1 unit of time after flying off).

Note that $\mathbf{c}'(t) = (e^t \cos(e^t), 1, -3t^2)$.

The tangent line at t_0 is parametrized as:

$\mathbf{l}(s) = \mathbf{c}(t_0) + s\mathbf{c}'(t_0) = (\sin e, 1, 3) + s(e\cos e, 1, -3)$.

Since we want the position of the particle 1 unit of time after flying off on the tangent, we get its position as:

$\mathbf{l}(1) = (\sin e, 1, 3) + (e\cos e, 1, -3) = (\sin e + e\cos e, 2, 0)$.

2.4.24:

Want to show that for the spiral $\mathbf{c}(t) = (e^t \cos t, e^t \sin t)$, the angle between $\mathbf{c}(t)$ and $\mathbf{c}'(t) = (e^t(\cos t - \sin t), e^t(\sin t + \cos t))$ is constant.

Recall, the angle θ between vectors $\mathbf{a}$ and $\mathbf{b}$ satisfies $\mathbf{a}\cdot\mathbf{b} = ||\mathbf{a}||\cdot||\mathbf{b}||\cos(\theta)$.

So the angle at time t between $\mathbf{c}(t)$ and $\mathbf{c}'(t)$ satisfies:

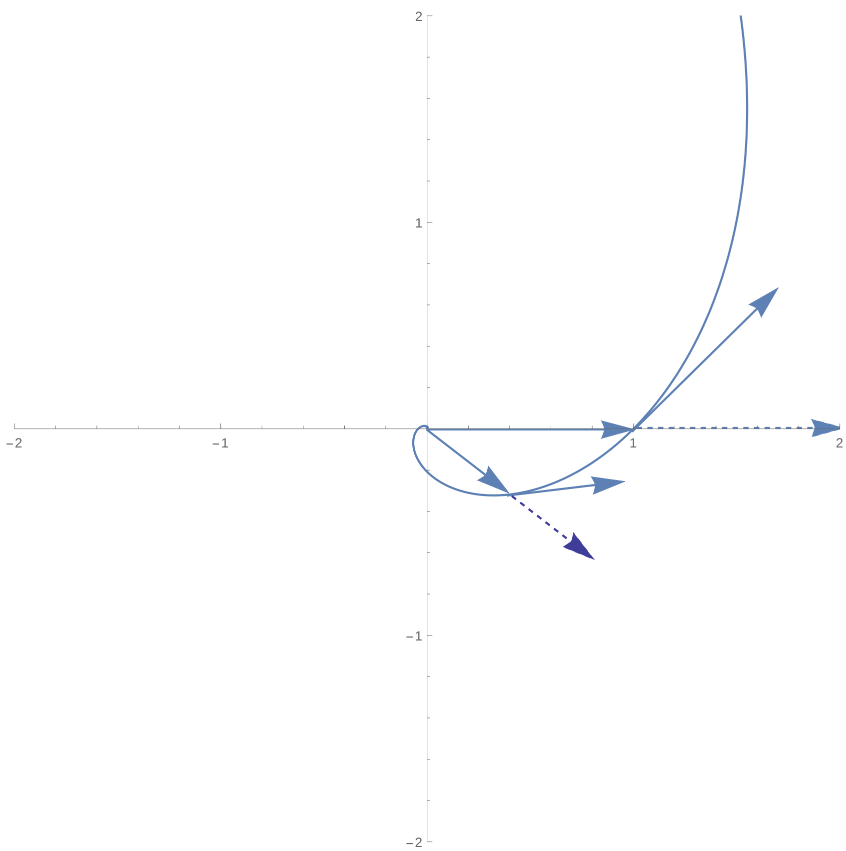
$$\cos(\theta(t)) = \frac{\mathbf{c}(t) \cdot \mathbf{c}'(t)}{\|\mathbf{c}(t)\| \cdot \|\mathbf{c}'(t)\|}$$

And since,
$\mathbf{c}(t) \cdot \mathbf{c}'(t) = e^{2t}(\cos^2 t + \sin^2 t) = e^{2t}$
$\|\mathbf{c}(t)\| = \sqrt{e^{2t}(\sin^2 t + \cos^2 t)} = e^t$
$\|\mathbf{c}'(t)\| = \sqrt{e^{2t}((\cos t - \sin t)^2 + (\cos t + \sin t)^2)} = e^t\sqrt{2(\cos^2 t + \sin^2 t)} = \sqrt{2}e^t$

We get:

$$\cos(\theta(t)) = \frac{e^{2t}}{\sqrt{2}e^t e^t} = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$$

So the angle between $\mathbf{c}(t)$ and $\mathbf{c}'(t)$, at any time t is the constant $\pi/4$, as depicted below:



2.5.3:

Recall first special case of the chain rule for $f \circ \mathbf{c}$:

$$(f \circ \mathbf{c})'(t) = (\nabla f)(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$$

(a) $f(x,y) = xy$, $\mathbf{c}(t) = (e^t, \cos t)$.

$(f \circ \mathbf{c})(t) = e^t \cos t$. Hence,
$(f \circ \mathbf{c})'(t) = e^t \cos t - e^t \sin t$.

Whereas, $(\nabla f)(\mathbf{c}(t)) = (y,x)|_{\mathbf{c}(t)} = (y,x)|_{(e^t, \cos t)} = (\cos t, e^t)$.
And $\mathbf{c}'(t) = (e^t, -\sin t)$, so we get:

$(\nabla f)(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = (\cos t, e^t) \cdot (e^t, -\sin t) = e^t \cos t - e^t \sin t$, the same as $(f \circ \mathbf{c})'(t)$, so the rule is verified.

(b) $f(x,y) = e^{xy}$, $\mathbf{c}(t) = (3t^2, t^3)$.

$(f \circ \mathbf{c})(t) = e^{3t^5}$. Hence,
$(f \circ \mathbf{c})'(t) = 15t^4 e^{3t^5}$.

Whereas, $(\nabla f)(\mathbf{c}(t)) = (ye^{xy}, xe^{xy})|_{\mathbf{c}(t)} = (ye^{xy}, xe^{xy})|_{(3t^2, t^3)} = (t^3 e^{3t^5}, 3t^2 e^{3t^5})$.
And $\mathbf{c}'(t) = (6t, 3t^2)$, so we get:

$(\nabla f)(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = (t^3 e^{3t^5}, 3t^2 e^{3t^5}) \cdot (6t, 3t^2) = 6t^4 e^{3t^5} + 9t^4 e^{3t^5}$, the same as $(f \circ \mathbf{c})'(t)$, so the rule is verified.

(c) $f(x,y) = (x^2 + y^2) \log \sqrt{x^2 + y^2} = \frac{(x^2+y^2)}{2} \log(x^2 + y^2)$, $\mathbf{c}(t) = (e^t, e^{-t})$.

$(f \circ \mathbf{c})(t) = \frac{1}{2}(e^{2t} + e^{-2t}) \log(e^{2t} + e^{-2t})$. Hence,
$(f \circ \mathbf{c})'(t) = (e^{2t} - e^{-2t}) \log(e^{2t} + e^{-2t}) + \frac{1}{2}(e^{2t} + e^{-2t}) \frac{2(e^{2t} - e^{-2t})}{(e^{2t} + e^{-2t})}$, i.e.

$(f \circ \mathbf{c})'(t) = (e^{2t} - e^{-2t})(\log(e^{2t} + e^{-2t}) + 1)$

$$
\begin{aligned}
(\nabla f)(\mathbf{c}(t)) &= \left(x\log(x^2+y^2) + \frac{(x^2+y^2)}{2}\frac{2x}{(x^2+y^2)}, y\log(x^2+y^2) + \frac{(x^2+y^2)}{2}\frac{2y}{(x^2+y^2)}\right)\Big|_{\mathbf{c}(t)} \\
&= (x\log(x^2+y^2)+x, y\log(x^2+y^2)+y)|_{(e^t,e^{-t})} \\
&= (e^t\log(e^{2t}+e^{-2t})+e^t, e^{-t}\log(e^{2t}+e^{-2t})+e^{-t}).
\end{aligned}
$$

And $\mathbf{c}'(t) = (e^t, -e^{-t})$, so we get:

$(\nabla f)(\mathbf{c}(t))\cdot\mathbf{c}'(t) = (e^t\log(e^{2t}+e^{-2t})+e^t, e^{-t}\log(e^{2t}+e^{-2t})+e^{-t})\cdot(e^t,-e^{-t}) = e^{2t}\log(e^{2t}+e^{-2t}) + e^{2t} - e^{-2t}\log(e^{2t}+e^{-2t}) - e^{-2t}$, indeed the same as $(f\circ\mathbf{c})'(t)$ upon rearrangement, so the rule is verified.

(d) $f(x,y) = xe^{x^2+y^2}$, $\mathbf{c}(t) = (t,-t)$.

$(f\circ\mathbf{c})(t) = te^{2t^2}$. Hence,
$(f\circ\mathbf{c})'(t) = e^{2t^2} + 4t^2e^{2t^2}$.

Whereas, $(\nabla f)(\mathbf{c}(t)) = (e^{x^2+y^2}+2x^2e^{x^2+y^2}, 2xye^{x^2+y^2})|_{(t,-t)} = (e^{2t^2}+2t^2e^{2t^2}, -2t^2e^{2t^2})$.
And $\mathbf{c}'(t) = (1,-1)$, so we get:

$(\nabla f)(\mathbf{c}(t))\cdot\mathbf{c}'(t) = (e^{2t^2}+2t^2e^{2t^2}, -2t^2e^{2t^2})\cdot(1,-1) = e^{2t^2}+4t^2e^{2t^2}$, the same as $(f\circ\mathbf{c})'(t)$, so the rule is verified.

2.5.4:

$\mathbf{c}'(t)$ was already computed in **2.5.3** as follows:

(a) $\mathbf{c}'(t) = (e^t, -\sin t)$

(b) $\mathbf{c}'(t) = (6t, 3t^2)$

(c) $\mathbf{c}'(t) = (e^t, -e^{-t})$

(d) $\mathbf{c}'(t) = (1,-1)$